PRINCIPAL BUNDLE STRUCTURES AMONG SECOND ORDER FRAME BUNDLES

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Abstract. Using a model for the bundle $\hat{\mathcal{F}}^2M$ of semi-holonomic second order frames of a manifold M as an extension of the bundle $\mathcal{F}^2M$ of holonomic second order frames of M, we introduce in $\hat{\mathcal{F}}^2M$ a principal bundle structure over $\mathcal{F}^2M$, the structure group being the additive group $A_2(n)$ of skew-symmetric bilinear maps from $\mathbb{R}^2 \times \mathbb{R}^n$ into $\mathbb{R}^n$. The composition of the projection of that structure with the existing projection of the bundle $\tilde{\mathcal{F}}^2M$ of non-holonomic second order frames of M over $\hat{\mathcal{F}}^2M$ provides a principal bundle structure in $\tilde{\mathcal{F}}^2M$ over $\mathcal{F}^2M$. These results close an existing gap in the theory of second order frame bundles.

1. Introduction

In the recent years, higher order frames have become an important tool that has contributed to the development of different chapters both inside the framework of differential geometry and physics. We can cite for example differential invariants, classical field theories, quantum field theory, continuum mechanics [1, 4, 12]. Since they were introduced by Ehresmann in 1995, non-holonomic and semi-holonomic higher order frame bundles have known several equivalent interpretations, most of them in terms of jet bundles. The characterization given here as a starting point is the one followed in the work by Elzanowski and Prishepionok [3], although later we make use of the interpretation of the non-holonomic and semi-holonomic second order frame bundles $\tilde{\mathcal{F}}^2M$ and $\hat{\mathcal{F}}^2M$ of a differentiable manifold M as extensions of the holonomic second order frame bundle $\mathcal{F}^2M$ [11].

One kind of problems that arises in the context of these theories is the structure problems. Recently, Brajerčík, Demko and Krupka [2] have realized a construction of a principal bundle structure on the r-jet prolongation of the linear frame bundle of an n-dimensional manifold; such prolongations involve semi-holonomic second order frames [8]. In this paper we also deal with a structure problem, raised in [10], for second order frames: the possible existence of a principal bundle structure on $\tilde{\mathcal{F}}^2M$ and $\hat{\mathcal{F}}^2M$ over $\mathcal{F}^2M$. The answer to the question is in the affirmative and the way in which we have treated it is a constructive way, in the sense that we have defined in detail such structures. The main drawback lies in finding a suitable projection $\hat{\pi}_2^2 \colon \hat{\mathcal{F}}^2M \to \mathcal{F}^2M$. An early work by Kolář [6] provides a symmetrization of semi-holonomic 2-jets using exact diagrams of vector bundles and splitting properties. Some other attempt of symmetrization in higher order using adittional geometric background has only reached a relative succeed [13]. The

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idea underlying in our definition of the projection $\hat{\pi}_2^2$ lies in the fact that the description of the bundle $\hat{\mathcal{F}}^2M$ as an extension of the bundle $\mathcal{F}^2M$ allows an algebraic manner to symmetrize the bilinear part of second order jets in a global way, on the basis that the symmetrization of a (semi-holonomic) jet somehow commutes with the composition with holonomic jets. The provided solution also completes the problem of the existence of principal bundle structures among the three second order frame bundles $\tilde{\mathcal{F}}^2M$, $\hat{\mathcal{F}}^2M$ and $\mathcal{F}^2M$, the linear frame bundle $\mathcal{F}M$, and the own manifold M. These relations are summarize in the Section 5 in the final part of the paper.

2. Preliminars

We are going to deal with five differentiable manifolds: an n-dimensional manifold M, its linear frame bundle $\mathcal{F}M$ and the bundles $\mathcal{F}^2M$, $\hat{\mathcal{F}}^2M$ and $\tilde{\mathcal{F}}^2M$ of holonomic, semi-holonomic and non-holonomic second order frames of M, respectively. Let us denote by π_0^1, π_0^2, $\hat{\pi}_0^2$ and $\tilde{\pi}_0^2$ the natural projections of $\mathcal{F}M$, $\mathcal{F}^2M$, $\hat{\mathcal{F}}^2M$ and $\tilde{\mathcal{F}}^2M$ on M and by $G^1(n) = GL(n,\mathbb{R})$, $G^2(n)$, $\hat{G}^2(n)$ and $\tilde{G}^2(n)$ their respective structure groups. The second order frames can be described as follows. Let $\tilde{\phi}\colon \mathbb{R}^n \to \mathcal{F}M$ be a differentiable map such that $\pi_0^1 \circ \tilde{\phi}$ is a diffeomorphism. The 1-jet $j_0^1\tilde{\phi}$ with source at the origin $0 \in \mathbb{R}^n$ and target in $\mathcal{F}M$ is a non-holonomic second order frame at the point $x = \pi_0^1(\tilde{\phi}(0)) \in M$. If $\tilde{\phi}$ satisfies the additional condition $\tilde{\phi}(0) = j_0^1(\pi_0^1 \circ \tilde{\phi})$, then $j_0^1\tilde{\phi}$ is called semi-holonomic. A holonomic second order frame is an invertible 2-jet $j_0^2 f$ where f is a differentiable map from $\mathbb{R}^n$ into M with source at $0 \in \mathbb{R}^n$ and target in M; a holonomic second order frame can be also seen as a semi-holonomic second order frame $j_0^1\tilde{\phi}$ verifying $\tilde{\phi} = (\pi_0^1 \circ \tilde{\phi})^{(1)} \circ \eta_1$, where $\eta_1\colon \mathbb{R}^n \to \mathcal{F}\mathbb{R}^n \equiv \mathbb{R}^n \oplus G^1(n)$ is the section given by $\eta_1 = \mathrm{id}_{\mathbb{R}^n} \times I$ and, if M_1 and M_2 are differentiable manifolds, $F^{(1)}\colon j_0^1 g \in \mathcal{F}M_1 \to j_0^1(F \circ g) \in \mathcal{F}M_2$ is the prolonged map of a local diffeomorphism $F\colon M_1 \to M_2$ between the corresponding linear frame bundles. In order to describe second order frames in local coordinates [7, 11], we put $\tilde{\phi}(r^a) \equiv (\phi^i(r^a), \phi^i_j(r^a))$, where $\{r^a\}$ are the natural coordinates in $\mathbb{R}^n$. Then the non-holonomic frame $j_0^1\tilde{\phi}$ is expressed as

$$\left(\phi^i(0), \phi^i_j(0), \frac{\partial \phi^i}{\partial r^j}(0), \frac{\partial \phi^k_l}{\partial r^j}(0)\right) \equiv (x^i, x^i_j, y^i_j, x^{kl}_j).$$

Here, (x^i, x^i_j) are the coordinates of $\tilde{\phi}(0)$ and (y^i_j, x^{kl}_j) are given by

$$\tilde{\phi}_*(0)\frac{\partial}{\partial r^j}(0) = y^i_j \frac{\partial}{\partial x^i}(z) + x^{kl}_j \frac{\partial}{\partial x^k_l}(z), \quad z = \tilde{\phi}(0), \quad j = 1, \ldots, n,$$

with (y^i_j) non-singular since $\pi_0^1 \circ \tilde{\phi}$ is a diffeomorphism. The semi-holonomic condition above results in $\phi^i_j(0) = \frac{\partial \phi^i}{\partial r^j}(0)$ (i.e., $x^i_j = y^i_j$) what translates into a system of local coordinates for $\hat{\mathcal{F}}^2M$ of the form (x^i, x^i_j, x^{kl}_j). On the other hand, the holonomic condition leads to $\frac{\partial \phi^k_l}{\partial r^j}(0) = \frac{\partial^2 \phi^k}{\partial r^l \partial r^j}(0)$, i.e., $x^{kl}_j = x^{kj}_l$.

Information of another models for non-holonomic and semi-holonomic second order frames can be found, for example, in [4, 9, 10, 11, 14].

Let $L_2(n)$ be the additive group of bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$ and let $S_2(n)$ and $A_2(n)$ be the additive subgroups of symmetric bilinear maps and skew-symmetric

bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$. Let us denote by $\{E_i\}$ the canonical basis in $\mathbb{R}^n$. If $f \in L_2(n)$ we define the bilinear maps f^t, f_s, $f_a \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $f^t(E_i, E_j) = f(E_j, E_i)$, $f_s = \frac{f+f^t}{2}$ and $f_a = \frac{f-f^t}{2}$; then $f_s \in S_2(n)$, $f_a \in A_2(n)$ and $f = f_s + f_a$. The groups $\tilde{G}^2(n)$, $\hat{G}^2(n)$ and $G^2(n)$ can be represented as the semi-direct product

$$\begin{aligned} \tilde{G}^2(n) &\equiv G^1(n) \times G^1(n) \times L_2(n), \\ \hat{G}^2(n) &\equiv G^1(n) \times L_2(n), \\ G^2(n) &\equiv G^1(n) \times S_2(n), \end{aligned}$$

with the group operations given by

$$(a, b, f)(a', b', f') = (aa', bb', a \circ f' + f(a', b'))$$

for $\tilde{G}^2(n)$ and

$$(a, f)(a', f') = (aa', a \circ f' + f(a', a'))$$

for $\hat{G}^2(n)$ and $G^2(n)$. For later use we state the following result, whose proof is straightforward.

Lemma 2.1. *If $a \in G^1(n)$ and $f \in L_2(n)$, then $(a \circ f)^t = a \circ f^t$, $(a \circ f)_s = a \circ f_s$, $(a \circ f)_a = a \circ f_a$ and $[f(a, a)]^t = f^t(a, a)$, $(f(a, a))_s = f_s(a, a)$, $(f(a, a))_a = f_a(a, a)$.*

In the last section we make an abridgement of the principal bundle relations among the five manifolds mentioned at the beginning of this section. Consequently, it is pertinent to recall that $\tilde{\mathcal{F}}^2M$, $\hat{\mathcal{F}}^2M$ and $\mathcal{F}^2M$ are all principal bundles over $\mathcal{F}M$ with projections $\tilde{\pi}^2_1$, $\hat{\pi}^2_1$ and π^2_1 carrying a jet $j^1_0\tilde{\phi}$, with $\tilde{\phi} \colon \mathbb{R}^n \to \mathcal{F}M$, into $\tilde{\phi}(0)$. The respective structure groups are $\tilde{G}^2_1(n)$, $L_2(n)$ and $S_2(n)$. The group $\tilde{G}^2_1(n)$ is $\tilde{G}^2_1(n) = G^1(n) \times L_2(n)$ endowed with the law $(a, f)(a', f') = (aa', f' + f(I, a'))$ [10].

3. The groups $\hat{G}^2(n)$, $G^2(n)$ and $A_2(n)$

In this section we are going to study some relations between the groups $\hat{G}^2(n)$, $G^2(n)$ and $A_2(n)$. First of all, let us note that the inverse $(a, f)^{-1}$ of $(a, f) \in \hat{G}^2(n)$ is $(a, f)^{-1} = (a^{-1}, -a^{-1} \circ f(a^{-1}, a^{-1}))$; if $(b, g) \in \hat{G}^2(n)$, the conjugation of (b, g) by (a, f) is given by

$$\begin{aligned} & (a, f) \cdot (c, g) \cdot (a, f)^{-1} = (a, f) \cdot (b, g) \cdot (a^{-1}, -a^{-1} \circ f(a^{-1}, a^{-1})) \\ = \ & (ab, a \circ g + f(b, b)) \cdot (a^{-1}, -a^{-1} \circ f(a^{-1}, a^{-1})) \\ = \ & (aba^{-1}, ab \circ [-a^{-1} \circ f(a^{-1}, a^{-1})] + ((a \circ g) + f(b, b))(a^{-1}, a^{-1}))) \\ = \ & (aba^{-1}, -a \circ b \circ a^{-1} \circ f(a^{-1}, a^{-1}) + a \circ g(a^{-1}, a^{-1}) + f(ba^{-1}, ba^{-1})). \end{aligned}$$

In particular, if $b = I$ we obtain

$$(a, f) \cdot (I, g) \cdot (a, f)^{-1} = (I, -f(a^{-1}, a^{-1}) + a \circ g(a^{-1}, a^{-1}) + f(a^{-1}, a^{-1})) = (I, a \circ g(a^{-1}, a^{-1}))$$

and the conjugation of the element $(I, g) \in \hat{G}^2(n)$ by $(a, f) \in \hat{G}^2(n)$ does not depend on f ([5]).

Lemma 3.1. *Let $(a, f) \in \hat{G}^2(n)$.*

i) *If $(I, h) \in G^2(n)$, the conjugation $(a, f) \cdot (I, h) \cdot (a, f)^{-1}$ belongs to $G^2(n)$.*

ii) *Let us consider the subgroup of* $\hat{G}^2(n)$ *defined by the semi-direct product* $G^1(n) \times A_2(n)$. *If* $(I,h) \in G^1(n) \times A_2(n)$, *then the conjugation* $(a,f) \cdot (I,h) \cdot (a,f)^{-1}$ *belongs to* $G^1(n) \times A_2(n)$.

iii) *There exist unique* $(b,g) \in G^2(n)$ *and* $(I,h) \in A_2(n)$ *such that* $(a,f) = (b,g)(I,h)$.

Proof:

Let us put

$$a(E_i) = a_i^j E_j \quad a^{-1}(E_i) = b_i^j E_j \quad h(E_i, E_j) = h_j^{li} E_l.$$

It follows that

$$\begin{aligned} (a \circ h(a^{-1}, a^{-1}))(E_i, E_j) &= a \circ h(a^{-1}E_i, a^{-1}E_j) \\ = a \circ h(b_i^r E_r, b_j^s E_s) &= a \circ (b_i^r b_j^s h(E_r, E_s)) \\ = a\big(b_i^r b_j^s h_s^{kr} E_k\big) &= b_i^r b_j^s h_s^{kr} a(E_k) = b_i^r b_j^s h_s^{kr} a_k^l E_l \end{aligned}$$

and if $a \circ h(a^{-1}, a^{-1})(E_i, E_j) = m_j^{li} E_l$, then

$$m_j^{li} = b_i^r b_j^s h_s^{kr} a_k^l.$$

Let us suppose that $(I,h) \in G^2(n)$, i.e., $h_s^{kr} = h_r^{ks}$. Then

$$m_i^{lj} = b_j^r b_i^s h_s^{kr} a_k^l = b_i^s b_j^r h_s^{kr} a_k^l = b_i^s b_j^r h_r^{ks} a_k^l = b_i^r b_j^s h_s^{kr} a_k^l = m_j^{li}$$

(note that in the second last equality we have interchanged the indexes r and s). That means that $a \circ h(a^{-1}, a^{-1}) \in G^2(n)$. This proves i). Statement ii) can be proved in the same way using that $(a,h) \in G^1(n) \times A_2(n)$ if and only if $h_k^{ij} = -h_j^{ik}$. In order to prove iii), only note that the elements (b,g) e (I,h) are $(b,g) = (a, f_s)$ and $(I,h) = (I, a^{-1} \circ f_a)$.

Statement iii) in Lemma 3.1 points out that $\hat{G}^2(n)$ can be also read as the semi-direct product $\hat{G}^2(n) \equiv G^2(n) \times A_2(n)$ in such a way that the law group in $G^2(n) \times A_2(n)$ springs from jet composition.

From now on, we shall often identify (I,h) with h; with this identification we obtain that $S_2(n)$ and $A_2(n)$ are closed subgroups of $\hat{G}^2(n)$ [5] and

Lemma 3.2. *The additive groups* $S_2(n)$ *and* $A_2(n)$ *are normal subgroups of* $\hat{G}^2(n)$.

Since $A_2(n)$ is a normal closed subgroup of $\hat{G}^2(n)$, then the quotient group $\frac{\hat{G}^2(n)}{A_2(n)}$ is a Lie group.

Proposition 3.3. $\frac{\hat{G}^2(n)}{A_2(n)}$ *and* $G^2(n)$ *are isomorphic Lie groups.*

Proof:

Define

$$\mu \colon (a,f)A_2(n) \in \frac{\hat{G}^2(n)}{A_2(n)} \to (a, f_s) \in G^2(n).$$

If $(a,f)A_2(n) = (a',f')A_2(n)$, there exists $(I,h) \in A_2(n)$ such that

$$\begin{array}{rl} & (a,f)(a',f')^{-1} = (a,f)(a'^{-1}, -a'^{-1} \circ f'(a'^{-1}, a'^{-1})) \\ = & (aa'^{-1}, -a \circ a'^{-1} \circ f'(a'^{-1}, a'^{-1}) + f(a'^{-1}, a'^{-1})). \end{array}$$

It follows then that $a = a'$ and $-f'(a^{-1}, a^{-1}) + f(a^{-1}, a^{-1}) \in A_2(n)$. Since a^{-1} represents an isomorphism, the last expression means that $-f' + f$ belongs to $A_2(n)$. Therefore $(-f'+f)^t = -(-f'+f) = f' - f$; hence $f^t + f = f'^t + f'$ and $f_s = f'_s$. So we have obtained $\mu((a,f)A_2(n)) = \mu((a',f')A_2(n))$ and μ is a well defined map. Moreover,

- $(a,f)A_2(n) \cdot (a',f')A_2(n) = (a,f)(a',f')A_2(n) = (aa', a \circ f' + f(a',a'))A_2(n)$;
- $\mu((a,f)A_2(n) \cdot (a',f')A_2(n)) = (aa', (a \circ f' + f(a',a'))_s)$;
- $\mu((a,f)A_2(n)) \cdot \mu((a',f')A_2(n)) = (a, f_s)(a', f'_s) = (aa', a \circ f'_s + f_s(a',a'))$.

Using Lemma 2.1 we obtain $a \circ f'_s + f_s(a',a') = (a \circ f' + f(a',a'))_s$ and μ is a group homomorphism. In order to prove that μ is injective, put $\mu((a,f)A_2(n)) = \mu((a',f')A_2(n))$. Then, $a = a'$ y and $f_s = f'_s$. According to the calculation developed above to show that μ is a well-defined map, it follows that proving $(a,f)A_2(n) = (a',f')A_2(n)$ is the same as proving $aa'^{-1} = I$ (what is true since $a = a'$) and $-f'(a^{-1},a^{-1}) + f(a^{-1},a^{-1}) \in A_2(n)$, i.e., $(-f'+f)(a^{-1},a^{-1}) \in A_2(n)$, or,

$$(-f'+f)(a^{-1},a^{-1})(E_i,E_j) = (f'-f)(a^{-1},a^{-1})(E_j,E_i).$$

Now then, since $f_s = f'_s$ we obtain $-f' + f = -f'_s - f'_a + f_s + f_a = -f'_a + f_a$, and therefore,

$$\begin{array}{rcl} (-f'+f)(a^{-1},a^{-1})(E_i,E_j) & = & (-f'_a + f_a)(a^{-1},a^{-1})(E_i,E_j) \\ = \ (-f'_a + f_a)(a^{-1}(E_i), a^{-1}(E_j)) & = & (f'_a - f_a)(a^{-1}(E_j), a^{-1}(E_i)) \\ = \ (f'_a - f_a)(a^{-1},a^{-1})(E_j,E_i) & = & (f'-f)(a^{-1},a^{-1})(E_j,E_i), \end{array}$$

and μ is injective. Finally, given $(b,g) \in G^2(n)$ and since $g = g_s$, there exists $(b,g)A_2(n) \in \frac{\hat{G}^2(n)}{A_2(n)}$ satisfying $\mu((b,g)A_2(n)) = (b, g_s) = (b,g)$, and μ is surjective.

There are several group structures on $G^1(n) \times L_2(n)$ in the literature. For example, the following two laws

$$\begin{array}{rcl} (a,f)(a',f') & = & (aa', a'^{-1} \circ f(a',a') + f') \\ (a,f)(a',f') & = & (aa', f + a \circ f'(a^{-1},a^{-1})), \end{array}$$

give both of them rise to groups which are isomorphic with $\hat{G}^2(n)$ [7].

The paper [2] by Brajerčík, Demko and Krupka deals with principal bundle structures in jet prolongations of higher order which include second order frames. There, the base manifold of the bundles is the manifold M. Libermann [8] proved that the first order jet prolongation $J^1\mathcal{F}M$ of the linear frame bundle $\mathcal{F}M$, which consists of 1-jets of local sections of $\mathcal{F}M$, can be identified with $\hat{\mathcal{F}}^2M$. The Lie group named $(T^r_nL^1_n, *)$ in [2] is the structure group of the principal bundle structure given there for the r-order jet prolongation $J^r\mathcal{F}M$. So it is not surprising that for $r = 1$ such structure group is isomorphic with $\hat{G}^2(n)$. The group law in $(T^1_nL^1_n, *)$ is described in [2] in local coordinates

as

$$(a^i_j, a^{ij}_k)(c^i_j, c^{ij}_k) = (a^i_m c^m_j, a^{il}_k c^l_j + a^i_l c^{lj}_m b^m_k),$$

where $(b^i_j) = (a^i_j)^{-1}$, and we have

Lemma 3.4. *The group law $*$ in $T^1_n L^1_n$ can be expressed as*

$$(a, f)(a', f') = (aa', f(a', I) + a \circ f'(I, a^{-1}))$$

and the group $(T^1_n L^1_n, *)$ is isomorphic with $\hat{G}^2(n)$ through the isomorphism of Lie groups

$$\tau\colon (a, f) \in T^1_n L^1_n \to \tau(a, f) = (a, f(I, a)) \in \hat{G}^2(n).$$

Proof:

If $a(E_i) = a^j_i E_j$, $a^{-1}(E_i) = b^j_i E_j$, $f(E_i, E_j) = a^{li}_j E_l$, $a'(E_i) = a^j_i E_j$ and $f'(E_i, E_j) = c^{li}_j E_l$, the description in local coordinates for $f(a', I) + a \circ f'(I, a^{-1})$ is given by

$$\begin{aligned}
& [f(a', I) + a \circ f'(I, a^{-1})](E_j, E_k) = f(a', I)(E_j, E_k) + a \circ f'(I, a^{-1})(E_j, E_k) \\
= \;& f(a'(E_j), E_k) + a \circ f'(E_j, a^{-1}(E_k)) = f(c^i_j E_i, E_k) + a \circ f'(E_j, b^i_k E_i) \\
= \;& c^i_j f(E_i, E_k) + a \circ (b^i_k f'(E_j, E_i)) = c^i_j a^{ri}_k E_r + a \circ (b^i_k c^{lj}_i E_l) \\
= \;& c^i_j a^{ri}_k E_r + b^i_k c^{lj}_i a(E_l) = c^i_j a^{ri}_k E_r + b^i_k c^{lj}_i a^r_l E_r \\
= \;& (c^r_j a^{ir}_k + b^r_k c^{lj}_r a^i_l) E_i
\end{aligned}$$

That means that the group law $*$ works as stated. Moreover we have

- $\tau((a, f)(a', f')) = \tau((aa', f(a', I) + a \circ f'(I, a^{-1})))$

  $= (aa', f(a', I)(I, aa') + a \circ f'(Ia^{-1})(I, aa')) = (aa', f(a', aa') + a \circ f'(I, a'))$
- $\tau(a, f)\tau(a', f') = (a, f(I, a))(a', f'(I, a')) = (aa', a \circ f'(I, a') + f(I, a)(a', a'))$

  $= (aa', a \circ f'(I, a') + f(a', aa'))$

and τ is an isomorphism. Of course, $\tau^{-1}(a, f) = (a, f(I, a^{-1}))$, and τ and τ^{-1} allow to recover the group law in $T^1_n L^1_n$ making $(a, f)(a', f') = \tau^{-1}(\tau(a, f)\tau(a', f'))$

4. Remaining bundle structures

Let $\mathcal{F}^2 M^{\hat{G}^2(n)} = \mathcal{F}^2 M \times_{G^2(n)} \hat{G}^2(n) = \frac{\mathcal{F}^2 M \times \hat{G}^2(n)}{\sim}$ be the quotient manifold obtained from $\mathcal{F}^2 M \times \hat{G}^2(n)$ and the equivalence relation

$$(p, k) \sim (p', k') \;\Leftrightarrow\; \text{there exists } \alpha \in G^2(n) \text{ such that } p' = p\alpha,\; k' = \alpha^{-1} k.$$

The bundle $[(p, k)] \in \mathcal{F}^2 M^{\hat{G}^2(n)} \to \pi^2_0(p) \in M$ is called the extension of $\mathcal{F}^2 M$ by the group $\hat{G}^2(n)$. The map

$$\vartheta\colon [(p, k)] \in \mathcal{F}^2 M^{\hat{G}^2(n)} \to \vartheta([(p, k)]) = pk \in \hat{\mathcal{F}}^2 M$$

is an isomorphism of principal bundles [11].

Let us define a projection $\hat{\pi}^2_2\colon \hat{\mathcal{F}}^2 M \to \mathcal{F}^2 M$ as

$$\hat{\pi}^2_2(pk) = \hat{\pi}^2_2(\vartheta([(p, k)])) = p \cdot (a, f_s), \quad \text{for } k = (a, f).$$

Before seeing that $\hat{\pi}_2^2$ is a well-defined map, we pose the following basic result, which shows the way in what composition with semi-holonomic jets and symmetrization commutes.

Proposition 4.1. *Let $(a, f) \in \hat{G}^2(n)$ and $(b, g) \in G^2(n)$. If $(b, g)(a, f) = (c, h)$, then $(b, g)(a, f_s) = (c, h_s)$.*

Proof:

Bearing in mind that $(b, g)(a, f) = (ba, b \circ f + g(a, a))$ and $(b, g)(a, f_s) = (ba, b \circ f_s + g(a, a))$, we have $h = b \circ f + g(a, a)$. The fact that $h_s = b \circ f_s + g(a, a)$ follows from $g = g^t$ and Lemma 2.1.

In order to see that $\hat{\pi}_2^2$ is a well-defined map, let $(p', k') \sim (p, k)$. Let $\alpha = (b, g) \in G^2(n)$ such that $p' = p\alpha,\ k' = \alpha^{-1}k$. We have that

$$\begin{aligned} & k' = \alpha^{-1}k = (b, g)^{-1}(a, f) = (b^{-1}, -b^{-1} \circ g(b^{-1}, b^{-1}))(a, f) \\ = \quad & (b^{-1}a, b^{-1} \circ f + (-b^{-1} \circ g(b^{-1}, b^{-1}))(a, a)) = (b^{-1}a, b^{-1} \circ (f - g(b^{-1}a, b^{-1}a))). \end{aligned}$$

Let us put $h = b^{-1} \circ (f - g(b^{-1}a, b^{-1}a))$. We must prove that

$$\hat{\pi}_2^2([(p', k')]) = \hat{\pi}_2^2([(p\alpha, \alpha^{-1}k)]) = p\alpha \cdot (b^{-1}a, h_s) = p \cdot (a, f_s) = \hat{\pi}_2^2([(p, k)]).$$

Since

$$(p\alpha) \cdot (b^{-1}a, h_s) = (p \cdot (b, g)) \cdot (b^{-1}a, h_s) = p \cdot ((b, g) \cdot (b^{-1}a, h_s))$$

it is enough to see that $(b, g) \cdot (b^{-1}a, h_s) = (a, f_s)$; but this follows from Proposition 4.1. So, $\hat{\pi}_2^2$ is a well-defined map. Moreover, $\hat{\pi}_2^2$ is surjective since given $q \in \mathcal{F}^2M$ we have $\hat{\pi}_2^2(q) = \hat{\pi}_2^2(q(I, 0)) = q \cdot (I, 0) = q$.

Lemma 4.2. $\hat{\pi}_0^2 = \pi_0^2 \circ \hat{\pi}_2^2$.

Proof:

Let us consider $pk \in \hat{\mathcal{F}}^2M$, with $p \in \mathcal{F}^2M$ and $k = (a, f) \in \hat{G}^2(n)$. Then

$$\hat{\pi}_0^2(pk) = \hat{\pi}_0^2(p) = \pi_0^2(p) = \pi_0^2(p(a, f_s)) = \pi_0^2(\hat{\pi}_2^2(pk)),$$

since $k \in \hat{G}^2(n)$, $(a, f_s) \in G^2(n)$ and the fibers over $\hat{\pi}_0^2(p) = \pi_0^2(p)$ in the bundles $\hat{\mathcal{F}}^2M \to M$ and $\mathcal{F}^2M \to M$ are, respectively, $p \cdot \hat{G}^2(n)$ and $p \cdot G^2(n)$.

Lemma 4.3. *Given $q \in \mathcal{F}^2M$ we have $(\hat{\pi}_2^2)^{-1}(q) = q \cdot A_2(n)$.*

Proof:

" $\subset$ " Let $p \in \mathcal{F}^2M$ and $k \in \hat{G}^2(n)$ be such that $pk \in (\hat{\pi}_2^2)^{-1}(q)$. Using Lemma 4.2 we obtain $\hat{\pi}_0^2(pk) = \pi_0^2(\hat{\pi}_2^2(pk)) = \pi_0^2(q)$. Therefore, pk and q lie in the same fiber in $\hat{\pi}_0^2 \colon \hat{\mathcal{F}}^2M \to M$ and there exists $\tilde{k} = (\tilde{a}, \tilde{f}) \in \hat{G}^2(n)$ verifying $pk = q\tilde{k}$. Consequently,

$$q(I, 0) = q = \hat{\pi}_2^2(pk) = \hat{\pi}_2^2(q\tilde{k}) = q \cdot (\tilde{a}, \tilde{f}_s)).$$

Since the action is free, we have $\tilde{a} = I$ and $\tilde{f}_s = 0$, i.e., $\tilde{a} = I$ and $\tilde{f} = -\tilde{f}^t$. Therefore, $(I, \tilde{f}) \in A_2(n)$ and we have proved that pk can be written in the form $q(I, \tilde{f})$ with $\tilde{f} = -\tilde{f}^t$ that is $pk \in q \cdot A_2(n)$.

" $\supset$ " Let us consider $h \in A_2(n)$. From the definition of $\hat{\pi}_2^2$ we obtain, since $h^t = -h$, that $\hat{\pi}_2^2(q(I,h)) = q(I, \frac{h+h^t}{2}) = q(I,0) = q$.

Lemma 4.4. $\hat{\pi}_1^2 = \pi_1^2 \circ \hat{\pi}_2^2$.

Proof:

Let us consider $pk \in \hat{\mathcal{F}}^2M$, with $p \in \mathcal{F}^2M$ and $k \in \hat{G}^2(n)$. Let us put $\hat{\pi}_2^2(pk) = q$. Therefore we can write $pk = q(I,h)$ with $(I,h) \equiv h \in A_2(n)$. Since $\hat{\pi}_1^2 \colon \hat{\mathcal{F}}^2M \to \mathcal{F}M$ is a principal bundle with $L_2(n)$ as structure group and $A_2(n) \subset L_2(n)$, then

$$\hat{\pi}_1^2(q(I,h)) = \hat{\pi}_1^2(q) = \pi_1^2(q),$$

since $\pi_1^2 = (\hat{\pi}_1^2)_{|\mathcal{F}^2M}$. Hence, $\hat{\pi}_1^2(pk) = \hat{\pi}_1^2(q(I,h)) = \pi_1^2(q) = \pi_1^2(\hat{\pi}_2^2(pk))$.

Let us also note that the projection $\hat{\pi}_2^2$ is differentiable since so is the map $f \in L_2(n) \to f_s \in S_2(n)$.

Lemma 4.5. *Let G be a Lie group. Let H_1 and H_2 be Lie subgroups of G verifying that for every element $g \in G$ there exist unique $h_1 \in H_1$, $h_2 \in H_2$ such that $g = h_1h_2$. Let M and N be differentiable manifold. Let $\hat{\xi} \colon M \to G$, $\xi \colon N \to H_1$ and $f \colon M \to N$ be differentiable maps. The map $\sigma \colon M \to H_2$ defined by $\hat{\xi}(x) = \xi(f(x))\sigma(x)$ is differentiable.*

Proof:

It is enough to notice that $\sigma(x) = (\xi(f(x)))^{-1}\hat{\xi}(x)$.

Theorem 4.6. *$\hat{\mathcal{F}}^2M$ is a principal bundle over $\mathcal{F}^2M$ with structure group $A_2(n)$ and projection $\hat{\pi}_2^2$.*

Proof:

The action on the right of the group $A_2(n)$ on $\hat{\mathcal{F}}^2M$ the we are going to consider is the restriction of the action on the right of $\hat{G}^2(n)$ on $\hat{\mathcal{F}}^2M$; so, such action is free.

We shall identify $\hat{\mathcal{F}}^2M$ with the image of $\mathcal{F}^2M^{\hat{G}^2(n)}$ by the map ϑ given above. Let us define

$$\Omega \colon pkA_2(n) \in \frac{\hat{\mathcal{F}}^2M}{A_2(n)} \to \Omega(pkA_2(n)) = p(a, f_s) = \hat{\pi}_2^2(pk)\mathcal{F}^2M$$

where $k = (a,f) \in \hat{G}^2(n)$.

The definition of Ω does not depend on the element on the equivalence class $pkA_2(n) = p(a,f)A_2(n) = \{(p(a,f))(I,g)\ ;\ g \in A_2(n)\}$. Indeed, $(p(a,f))(I,g) = p((a,f)(I,g)) = p(a, a \circ g + f)$; so, using Lemma 2.1 and $g = -g^t$, we have

$$\begin{aligned} \Omega \quad & (pk(I,g)A_2(n)) = \Omega(p(a, a\circ g + f)A_2(n)) \\ = \quad & p\left([a \circ g + f]_s\right) = p(a, f_s) = \Omega(pkA_2(n)) \end{aligned}$$

Now, let $pkA_2(n)$, $p'k'A_2(n) \in \frac{\hat{\mathcal{F}}^2M}{A_2(n)}$ be such that $\Omega(pkA_2(n)) = \Omega(p'k'A_2(n)$. Then $\hat{\pi}_2^2(pk) = \hat{\pi}_2^2(p'k')$, so that pk and $p'k'$ lie in the same fiber of the projection $\hat{\pi}_2^2 \colon \hat{\mathcal{F}}^2M \to \mathcal{F}^2M$; by Lemma 4.3 there exists $g \in A_2(n)$ such that $pk(I,g) = p'k'$, i.e., $pkA_2(n) =$

$p'k'A_2(n)$ and Ω is injective. Ω is also surjective since, given $q \in \mathcal{F}^2M$ there exists $qA_2(n) \in \frac{\hat{\mathcal{F}}^2M}{A_2(n)}$ such that $\Omega(qA_2(n)) = \hat{\pi}^2_2(q) = q$. Let us finally note that Ω and Ω^{-1} are differentiable since so is $\hat{\pi}^2_2$.

Hence we have proved that $\mathcal{F}^2M$ is the quotient space of $\hat{\mathcal{F}}^2M$ by the equivalence relation induced by $A_2(n)$. Moreover, if $\pi \colon p \to \pi(p) = [p]$ is the canonical projection, then $\Omega \circ \pi = \hat{\pi}^2_2$.

Next, let $V \subset M$ be an open set and let $\Xi \colon (\pi^2_0)^{-1}(V) \to V \times G^2(n)$ and $\hat{\Xi} \colon (\hat{\pi}^2_0)^{-1}(V) \to V \times \hat{G}^2(n)$ be trivializations for the bundles $\pi^2_0 \colon \mathcal{F}^2M \to M$ and $\hat{\pi}^2_0 \colon \hat{\mathcal{F}}^2M \to M$, respectively, that we choose in such a way that $\hat{\Xi}(q) = \Xi(q)$ for every $q \in (\pi^2_0)^{-1}(V)$. Let us put $U = (\pi^2_0)^{-1}(V)$. From Lemma 4.2 we obtain $(\hat{\pi}^2_0)^{-1}(V) = (\hat{\pi}^2_2)^{-1}(\pi^2_0)^{-1}(V) = (\hat{\pi}^2_2)^{-1}(U)$. Define

$$\Sigma \colon p \in (\hat{\pi}^2_2)^{-1}(U) \to \Sigma(p) = (\hat{\pi}^2_2(p), \sigma(p)) \in U \times A_2(n)$$

where $\sigma \colon (\hat{\pi}^2_2)^{-1}(U) \to A_2(n)$ is given as follows: if $\Xi(q) = (\pi^2_0(q), \xi(q))$ and $\hat{\Xi}(p) = (\hat{\pi}^2_0(p), \hat{\xi}(p))$, then

$$\hat{\xi}(p) = \xi(\hat{\pi}^2_2(p))\sigma(p).$$

Using Lemma 4.5 we have that σ, and Σ, are differentiable.

Let $p \in (\hat{\pi}^2_2)^{-1}(U)$. Since $(\hat{\pi}^2_2)^{-1}(\hat{\pi}^2_2(p)) = \hat{\pi}^2_2(p)A_2(n)$ (see Lemma 4.3), then we can put $p = \hat{\pi}^2_2(p)(I, h)$, with $h \in A_2(n)$. Therefore

$$\xi(\hat{\pi}^2_2(p))(I, h) = \hat{\xi}(\hat{\pi}^2_2(p))(I, h) = \hat{\xi}(\hat{\pi}^2_2(p)(I, h)) = \hat{\xi}(p).$$

Comparing the definition of σ with the former expression and keeping in mind Lemma 3.1, iii), we obtain

$$\sigma(p) = (I, h).$$

Moreover, $(\hat{\pi}^2_2)^{-1}(q) = qA_2(n)$ for every $q \in \mathcal{F}^2M$; therefore Σ is a bijection and, as a consequence, Σ is a diffeomorphism.

Finally we obtain that the map σ verifies $\sigma(p(I, \tilde{h})) = \sigma(p)(I, \tilde{h})$ for every $\tilde{h} \in A_2(n)$ and $p = \hat{\pi}^2_2(p)(I, h) \in (\hat{\pi}^2_2)^{-1}(U)$. Of course, we should see that $\sigma(p(I, \tilde{h})) = (I, h)(I, \tilde{h})$, but this is immediate from the very definition of σ and from the fact that

$$\xi(\hat{\pi}^2_2(p(I, \tilde{h})))(I, h)(I, \tilde{h}) = \xi(\hat{\pi}^2_2(p))(I, h)(I, \tilde{h}) = \hat{\xi}(p)(I, \tilde{h}) = \hat{\xi}(p(I, \tilde{h})).$$

This completes the proof.

In [10] it is proved that $\tilde{\mathcal{F}}^2M$ can be endowed with a $G^1(n)$-principal bundle structure over $\hat{\mathcal{F}}^2M$. Following the lines in the proof of Theorem 5.2 in [10], straightforward calculations show that the projection $\pi \colon \tilde{\mathcal{F}}^2M \to \hat{\mathcal{F}}^2M$ defined there is given in local coordinates by $\pi(x^i, x^i_j, y^i_j, x^{kl}_j) = (x^i, x^i_j, x^{kl}_r x^r_j)$ (in other words, $\pi(x, a, b, f) = (x, a, f(I, a))$), and that the action on the right of the group $G^1(n)$ on $\tilde{\mathcal{F}}^2M$ is expressed, identifying $l \in G^1(n)$ with $(I, l, 0) \in \tilde{G}^2(n)$, as $(x, a, b, f)(I, l, 0) = (x, a, bl, f(I, l))$. This action is, in fact, the restriction of the action on the right of $\tilde{G}^2(n)$ on $\tilde{\mathcal{F}}^2M$.

Now we must identify $h \in A_2(n)$ with $(I, I, h) \in \tilde{G}^2(n)$. Let us consider the subgroup of $\tilde{G}^2(n)$ given by the semi-direct product $\tilde{G}^2_2(n) = G^1(n) \times A_2(n)$, where the product law is given by

$$(I, l, h)(I, l', h') = (I, ll', h' + h(I, l')).$$

Notice to warnings: first, if $(I, l, 0) \in G^1(n)$ and $(I, I, h) \in A_2(n)$, then we obtain $(I, l, 0)(I, I, h) = (I, l, h)$; second, this group $\tilde{G}^2_2(n)$ has nothing to do with the group $G^1(n) \times A_2(n)$ of Lemma 3.1, ii), which is a subgroup of $\hat{G}^2(n)$.

Theorem 4.7. *$\tilde{\mathcal{F}}^2 M$ is a principal bundle over $\mathcal{F}^2 M$ with structure group $\tilde{G}^2_2(n)$ and projection $\tilde{\pi}^2_2 = \hat{\pi}^2_2 \circ \pi$.*

Proof:

The action on the right of the group $\tilde{G}^2_2(n)$ on $\tilde{\mathcal{F}}^2 M$ is the restriction of the action on the right of $\tilde{G}^2(n)$ on $\tilde{\mathcal{F}}^2 M$; so, such action is free. This action can be described as

$$(x, a, b, f)(I, l, h) = (x, a, bl, a \circ h + f(I, l)),$$

an expression that can be obtained as well as $((x, a, b, f)(I, l, 0))(I, I, h)$. Let us also point out that the group law in $\tilde{G}^2_2(n)$ is identical to the law of the group $\tilde{G}^2_1(n)$ introduced at the end of Section 2, so the fact that $p((I, l, h)(I, l', h')) = (p(I, l, h))(I, l', h')$ for every $p \in \tilde{\mathcal{F}}^2 M$ and (I, l, h), $(I, l', h') \in \tilde{G}^2_2(n)$ follows from the fact that $\tilde{\mathcal{F}}^2 M$ is a $\tilde{G}^2_1(n)$-principal bundle over $\mathcal{F}M$.

The rest of the proof is immediate.

It is also obvious, from the definition of $\tilde{\pi}^2_2$ itself, that the projection $\tilde{\pi}^2_2$ satisfy the appropriate relations of commutativity with all the other projections.

5. Conclusion

Theorems 4.6 and 4.7 provide the bundle structures missing in [10] and conclude the relations of existence of principal bundle structures among the five manifolds $\tilde{\mathcal{F}}^2 M$, $\hat{\mathcal{F}}^2 M$, $\mathcal{F}^2 M$, $\mathcal{F}M$ and M we are dealing with. We collect all of them in the following picture, where we have pointed out at the right side of the arrows the corresponding structure group:

descending to the first level

$$\begin{array}{cccc} \tilde{\mathcal{F}}^2 M & \hat{\mathcal{F}}^2 M & \mathcal{F}^2 M & \mathcal{F}M \\ \downarrow \tilde{G}^2(n) & \downarrow \hat{G}^2(n) & \downarrow G^2(n) & \downarrow G^1(n) \\ M & M & M & M \end{array}$$

descending to the second level

$$\begin{array}{ccc} \tilde{\mathcal{F}}^2 M & \hat{\mathcal{F}}^2 M & \mathcal{F}^2 M \\ \downarrow \tilde{G}^2_1(n) & \downarrow L_2(n) & \downarrow S_2(n) \\ \mathcal{F}M & \mathcal{F}M & \mathcal{F}M \end{array}$$

descending to the third level

$$\begin{array}{cc} \tilde{\mathcal{F}}^2 M & \hat{\mathcal{F}}^2 M \\ \downarrow \tilde{G}^2_2(n) & \downarrow A_2(n) \\ \mathcal{F}^2 M & \mathcal{F}^2 M \end{array}$$

$$\begin{array}{cc} & \tilde{\mathcal{F}}^2 M \\ \textit{descending to} \\ \textit{the fourth level} & \downarrow G^1(n) \\ & \hat{\mathcal{F}}^2 M \end{array}$$

For a detailed description of the principal bundle structures not treated here, see [10] and references therein.

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